From One to Infinity: Historical Development and Student View of Large Numbers

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The importance of a consideration of large numbers in primary and early secondary school should not be underestimated. Historically, in Indian mathematics (and Mayan), a study of large numbers seems to have provided the impetus for the development of a place value number system. Present day students do not have to create a number system, but they do need to understand its structure in order to develop number sense and operations. We believe that this can be done more quickly through reflection on large numbers. We consider different types of large numbers in use in India prior to the construction of the present number system and examine Year 9 student responses to a questionnaire on large numbers. These are categorised and the results suggest that many students show competence in naming and using large numbers, and that some are in the process of learning beyond their curriculum level.

Background

Numbers and counting are a part of everyone's life, and understanding numbers and the number system structure is fundamental to progress in mathematics, particularly in arithmetic and algebra. A rich understanding of the number system implies a sound grasp of its multiplicative structure, which, in turn, is essential for understanding operations on numbers. A growing body of research literature on the subject, gives evidence that many students still find it difficult to comprehend the structure of the number system (e.g., Thomas, 2004). In recent years, many researchers and educators (e.g., Fauvel & van Maanen, 2000; Tzanakis & Arcavi, 2000) have turned to the history of mathematics in an attempt to understand student difficulties and inform practice, suggesting an integration of history into teaching and learning. The theoretical framework for the use of historical analysis in contemporary research is sometimes essentially that ontogeny recapitulates phylogeny, that is the individual's mathematical development may pass through the same successive stages, although in an abbreviated form, as the human species did in its development of mathematical ideas (see for e.g., Sfard, 1995; Radford, 2000). Hence there is much to be gained from a knowledge of the historical development, such as the order in which concepts arose and why that was the case. Furthermore, as Tzanakis and Arcavi (2000) elaborate, although mathematics is taught in a deductively oriented format, history shows that this format comes only after mathematics reached maturity. While this is necessary to avoid long-winded accounts, as well as questions and problems that constituted the motivation for the development of an idea, as well as doubts, twists and turns, false paths, dead-ends etc, these all remain hidden under a linearly organized body of knowledge. While, teaching of mathematics should not necessarily follow the same complicated historical route, there will be lessons to be learned from history that may suggest possible ways to present the subject rather than using a strictly deductive, or an *ad hoc,* introduction of concepts. One lesson from history, according to Sfard (1995, pp. 15, 16) is the role of cognitive dissonance, since due to "the inherent properties of knowledge itself,…difficulties experienced by an individual learner at different stages of knowledge construction may be quite close to those that once challenged generations of mathematicians". In summary, historical awareness may be helpful in adopting appropriate teaching strategies for the classroom and also in providing us with a toolkit to understand student behaviours (Radford, 2000). However, such tasks according to Radford, require a theoretical framework that includes an explicit epistemological stance and a clear articulation of the historical, psychological and methodological domains. The research presented here is founded on the insights and hypotheses from history described above, and begins the process of applying a possible theoretical framework to number system learning.

A study of the history of the development of the present number system reveals that it originated in India and was then carried to Europe by the Arabs (Datta & Singh, 2001). Hence this research concentrates on Indian developments. One of the many crucial mathematical developments in India preceding the perfection of the present place value number system, and which proved of particular significance, was the consideration of large numbers. The ancient Indian fascination with these large numbers, like the Maya, was principally due to their interest in time scales and astronomy, which were important in order to draw up a calendar and to determine the time and place for rituals.

Historical Perspectives

Indian texts reveal a number of descriptions of large numbers, which are outlined below.

1. In the *Yajurveda Samhita* (c.2000 BC) of the Vedas (in Sanskrit verse), the following list of number names is given: *Eka* (1), *Dasa* (10), *Sata* (10^2), *Sahasra*(10^3), *Ayuta* (10^4), *Niyuta* (10^5), *Prayuta* (10^6), *Arbuda* (10^7), *Nyarbuda* (10^8), *Samudra* (10^9), *Madhya* (10^{10}), *Anta* (10^{11}), *Parardha*(10^{12}). We note that each of these named denominations is 10 times the preceding one, so that they were aptly called the *dasagunottara samjna* (decuple terms) confirming that there was a definite systematic mode of arrangement in the naming of numbers. In *Taittriya Samhita,* this list was extended to *loka* (10^{19}) (Datta & Singh, 2001).

2. In the Buddhist work *Lalitavistara* (100 B.C.E), there are examples of series of number names based on the centesimal scale. For example, the mathematician Arjuna asks how the counting would go beyond *koti* (10^7) on the centesimal scale, and Bodhisattva replies: Hundred *kotis* are called *ayuta* (10^9), hundred *ayutas* is *niyuta* (10^{11}), hundred *niyutas* is *kankara*(10^{13}),…and on to *sarvajna* (10^{49}), *vibhutangama* (10^{51}), *tallaksana* (10^{53}). In this way 10^{53} became part of a series that went up to the enormous number 10^{421}! (Gupta, 1987).

3. In the epic *Ramayana,* written by *Valmiki*, there is a mention of the size of *Rama's* army as being $N = 10^{10}+ 10^{14}+ 10^{20}+ 10^{24}+ 10^{30}+ 10^{34}+ 10^{40}+ 10^{44}+ 10^{52}+ 10^{57}+ 10^{62}+ 5$ (Joseph, 2000). Even though these numbers seem fantastic, the fact that names for such numbers existed indicates that the Vedic Indians were quite at home with very large numbers.

4. Another interesting series of number names increasing by multiples of 10 million is found in *Kaccayana's* Pali Grammar. Interestingly, the number names go up to *koti* (10 million) in multiples of 10 and then by ten millions. It says that: hundred-hundred thousand *kotis* (10^7) give *pakoti* (10^{14}), hundred-hundred thousand *pakotis* is *kotipakoti* (10^{21}), similarly we have *nahuta* (10^{28}),…...*kathana* (10^{126}), *mahakathanas*, *asankhyeya* (10^{140})! (Datta & Singh, 2001).

5. In the Vedic literature, time is reckoned in terms of *yugas*. The four *yugas* are *Satya-yuga*, *Treta yuga*, *Dwapara yuga* and *Kali yuga*. According to Hindu cosmology, the time-span of these four *yugas* is 1728000, 1296000, 864000, and 432000 years, respectively, in the ratio 4:3:2:1. The total of these four *yugas* was considered as one *yuga*-cycle or *Mahayuga* and was thus 4320000 years (Srinivasaiengar, 1967). Moreover, it is believed that 1000 such *yuga*-cycles comprise one day in the life of *Brahma* which is 4,320,000,000 years and one day & night period is 8.64 billion years, this is further extended to 311 trillion (10^{12}) years.

6. In the *Anuyogadvara-sutra* (c.100 BCE), a Jaina text, the number of human beings in the world is given as 2^{96}. It is also in this work that the first mention of the word 'place' is used for a denomination. Other large number examples are $(8400000)^{28}$ and 256^{256}! (Gupta, 1987).

The Concept of Infinity

While the numbers described above are often extremely large it is interesting to note that there was also an early idea of infinity. The peace song in the *Isa* Upanishad of the Vedas (Gupta, 2003) mentions *purna* which is taken to mean 'infinite fullness' (interestingly *purna* was also a word-numeral used for zero). The Jains (600 BCE) had an interest in very large numbers and infinity and they classified all numbers into three groups, namely *Enumerable*, *Innumerable*, *Infinite*, and these were further subdivided into three orders. In the third group, Joseph (2000) describes how Jaina mathematics recognized five different kinds of infinity: *infinity in one direction; infinity in two directions; infinite in area; infinite everywhere; and infinite perpetually*. Also, Joseph mentions that the Jains were the first to discard the idea that all infinities are the same, an idea held in Europe until Georg Cantor's work in the late 19th century delineated countable infinities, such as $\aleph_0$, from the uncountable.

As can be seen, enormous numbers were dealt with from a very early time period in India and it is significant to note the extent to which these numbers were taken. The handling of such large numbers (at first in the oral tradition) probably spurred the development of the decimal number system with place value and zero. Several stages involving symbolization, concept of place value and zero were crossed before the final construction of the number system (Datta & Singh, 2001). A detailed discussion of this is beyond the scope of this paper. However, one significant aspect in terms of pedagogy is that in the naming of large numbers, the multiplicative structure (particularly exponential) was already present from early Vedic times.

Large Numbers and Theoretical Perspectives

In today's world of space travel, computer technology and huge government budgets, large numbers are relatively common. Students meet such numbers in their everyday life and in their secondary schooling. For example, Avogadro's number in chemistry ($6.022D10^{23}$) and the use of exponential functions, with large values, to model many scientific and social phenomena. Some other examples are disease prediction, radioactive waste and national debt that involve very large (or very small) numbers and exponential growth/decay.

In this regard the New Zealand Curriculum (2007) recommends that in the 6^{th} and 7^{th} year of schooling (level 3), students' number knowledge needs to include 'how many tenths, tens, hundreds and thousands are in whole numbers' and in the 8^{th} and 9^{th} year of schooling (Level 4) students should 'know the relative size and place value structure of positive and negative integers and decimals to three places'. It does not specifically mention naming of large numbers at these levels or at the lower levels, in contrast with the Astronomy section of the Science Curriculum that states that students should 'Investigate the components of the solar system, developing an appreciation of the distances between them'. There has been on-going curriculum debate over the years (Irwin & Burgham, 1992) on whether one should only employ numbers that students can use and understand. This reflects a concern of curriculum designers to make sure that students have a sound understanding of small numbers, their relationship to one another, and the related operation facts, before moving on to larger numbers. While this is important, Irwin and Burgham (*ibid*) suggest that children know such words as 'hundred', 'thousand' and 'million' exist, that these are number names, and that they stand for large numbers. Hewitt (1998) agrees and says that naming different numbers should be the first step before shifting to operations and structure, multiple written representations and some 'important arithmetic'. Young children may gradually become aware of the structure these numbers conform to by assimilating different concepts involving them. In particular they may see the need for compression of the large numbers into smaller cognitive units (Barnard & Tall, 1997) and from there the exponential form, and hence place value. However, as noted by Hewitt (1998), understanding the meaning of numbers takes longer than necessary for students as many teachers think that number names should be taught in numerical order. The 'Universal Number Chart' developed by Reddy and Srinath (2001), which is similar to Hewitt's 'Tens chart', stresses the structure of numbers through language. Reddy and Srinath suggest the chart may guide children to construct large numbers and state that their approach is an easy, fast and better technique to introduce numbers to students than traditional means, increasing students' interest, motivation and understanding. The argument of Zazkis (2001) is that student contemplation of large numbers may help them construct a sense of form and structure. She found that students who experienced difficulties with expressing generality were able to notice structure (rather than computing the number) and express this structure in algebraic notation. Hence reflection on large numbers may help students not only to understand number structure and operations but may also lead them to a development of some algebraic language.

Given the importance of understanding number, and particularly very large numbers, it is a little surprising that few studies appear to have highlighted students' experience and understanding of large numbers in all their complexity. In view of this and also in view of the role that large numbers played in the historical development of the number system, this small-scale study undertook an initial investigation of students' notions of large numbers in order to develop a framework that might inform teaching practice.

Method

The subjects in this research study comprised a Year 9 class of 13 year-olds in a multicultural secondary school in Auckland, New Zealand. The class represented a variety of cultural backgrounds including Korean, Chinese, Zimbabwean, Indian, Cypriot, Maori, Pacific Island and New Zealand European students. However, many of the students had their Intermediate schooling in New Zealand and hence were proficient in English. Five students were attending ESOL classes. The class had had some intermittent previous experience of large numbers and exponential notation (such as $7\times7\times7\times7 = 7^4$) during the year, but had no specific teaching on the topic. The questionnaire (see Figure

1) was given in December, in the last term of the school year, and only 22 students of the 27 students in the class were present on the day.

Results and Discussion

One of the main purposes of this initial study was to understand student notions of large numbers and to attempt to categorize them. An analysis of the results would thus provide us with a toolkit to assist with understanding of student difficulties, number activities and levels of knowledge. The results would also help develop a framework to inform teaching and learning. The second purpose was to identify areas that might be useful for future research. In the event, the answers to the questions could be categorized in five ways: different ways of writing large numbers; methods of computation with large numbers; production of questions incorporating large numbers; use of exponentiation; and the emerging concept of infinity.

Some students found the questionnaire difficult and hence S11, S14, S16, S17 and S22 did not attempt many of the questions. Perhaps these students needed more systematic practice at naming very large numbers before shifting their focus to operations and structure. As happened historically in India, the naming of large numbers occurred for many centuries before the final development of the place value system, and these students' ontogeny may need to mirror this. Students answered these questions the week after the end of the year exams, and two of the students, S16 and S3, said that they were too tired after the exams to answer any of the questions. Out of the five ESOL students one was absent and two (S14 and S22) of the remaining four students were hindered by language difficulties and hence answered very few questions. In fact, S14 was very new to the class and had arrived only a couple of days before. This class was also participating in another research project where they were using computer algebra system (CAS) calculators, and so all of them had a CAS calculator to use. However, on the day of the questionnaire only some students brought it to class; others used a scientific calculator, and still others did not use one at all. This may have been a factor in responses too.

Answer the following questions as best as you can

1. What is the largest number that you can say in five seconds?
2. What is the largest number that you know?
3. Suppose you are asked to count to the number in 2 above, what would be your method?
4. If you are to count faster and faster to the number in 2 above, what would be your method then?
5. The following are two examples of very large numbers taken from an ancient text.
   a. A rajju is the distance travelled by a bird in six months if it covers a hundred thousand yojana (approximately a million kilometers) in each blink of its eye.
      (i) Work out this number as best as you can
      (ii) Can you think up a similar example of a large number, but which is more relevant to the present?
   b. A palya is the time it will take to empty a cubic vessel of side one yojana (10 kilometres) filled with the wool of new-born lambs if one strand is removed every century.
      (i) Work out this number as best as you can.
      (ii) Think up a similar example and write it down.
6. What is infinity?
7. Can you think of a situation where very, very large numbers need to be used?
8. Comments:

n.b. Space was provided for answers.

Fig. 1. The questions used in the study.

Different Ways of Writing Large Numbers

All students except one were able to give examples of large numbers. The answers varied in size and format, with some writing them as numerals, others in words or number names and some as a mix of words and numerals. Some examples were: 999 999 999 999 999 (written by S10); 1 followed by 33 zeros (S21); 123456789 (S18); 10^{11} (S4); and others showed a knowledge of number names, writing googol (S22), googolplex (S11) or 1 decillion (S12). There was also a mixture of words and numerals from some, with S15 giving 999 decillion 999 nonillion 999 octillion…and so on up to 999 thousand and 999. This is interesting in that she wrote 9 into every place, thereby seeking to maximize the value of the number. S9's answer was similar. Her number had 102 places/digits and she wrote 9 in all of them. The difficulty with the use of number names is that we cannot be sure that those who wrote googol and googolplex actually knew the value of them. Also, it is interesting that when asked to write the largest number that you can say in five seconds, one student (S4) wrote infinity, both as a word and as a symbol. For Q3, which was on counting to a large number, S20 wrote "its impossible to say" and then explained; "The reason I didn't say anything because my largest number is infinite. I can keep on going" (see below for a fuller discussion of infinity).

Can Students Compute with Large Numbers?

Question 5 sought to examine whether students could not only write, but could also work with large numbers, and nine of the 22 students attempted to do so. Although a couple of students seemed to have guessed the answer, others provided answers that varied depending on the method they employed to calculate the number of seconds in six months. S4 and S5 both showed a similar method of calculation, using 1000000×60×60×24×182(.5) and also worked out a correct answer, using their knowledge of multiplying by powers of 10 to write the answer in full decimal form, such as 15,724,800,000,000. S20 used a similar method but did not work out the final answer, instead writing "60×60×24×7×4×6× distance". S9 and S18 not only showed the correct working they also converted their answer in the standard form, S9 giving 1.5812×10^{13}, and S18 as 1.58112E13. This class had not been taught standard form and so this latter working was probably due to the use of CAS calculators. In fact S9 asked the teacher what 1.58112E13 meant, and when this was explained to her she wrote her answer in the correct scientific form.

Can Students Produce Questions That Use Large Numbers?

It is one thing to be able to use large numbers but it is quite another to generate them. Question 5b (ii) asked the students to think up an example of their own involving large numbers, producing a range of interesting answers. While most of those who replied had solutions involving the multiplication of just two large numbers, among the answers were: "If we took 10000 steps a day how many would we take in a decillion years" (S21); "If there are 837 394 506 320 097 891 023 CO_2 molecules in the world and we are adding 7 trillion each month, how many CO_2 molecules will there be in a century" (S13); and "If there was 7 000 000 000 people in the world today and 100 are born every second, how many would be born in 10 years?" (S2). These gave evidence of the ability to see a problem with a multiplicative solution of more than two steps.

Evidence of Exponentiation

Use of exponentiation is a crucial step on the way to number system construction, and there was some evidence of numbers written in exponential form. For example, as described above, S9 gave her answer to 5a(i) in exponential form as 1.5812×10^{13}. Similarly, others used exponential form, with S18 using 10^6 for a million, and S6 giving the largest number he knew as 10^{33} (in Q2). S19's answer showed her advanced exponential thinking when she wrote $(10^{100})^{10}$, making her one of only two students to write her number as the power of a power, and although S5 did not use exponential notation she still displayed knowledge of repeated multiplication such as 100×100×100…12 times. S8 wrote his answer in repeated exponential form, using $(999999999)^{(999999)^{(9999999)}}$, and to two more powers, apparently aware that writing the number in exponential form can rapidly increase the value of the number. However, it is unlikely he has understood the enormous size of the number created.

Students' Ideas of Infinity

For students of this age, the concept of even countable infinity,, is a difficult one. However, some of these students did have some properties of the concept, writing "there is no largest number", "there is no ending to a number" (both by S18); "infinity is a series of numbers that will never end, it is never ending. is the symbol of infinity" (S15), "infinity is a word used to say that a number is constant, on going on going and going and going" (S13), "A number that keeps going and going and going……forever" (S19), "that means there is lots of numbers and it just keeps going" (S2), "It (infinity) is always continued" (S1), "It's an unlimited number that never ends, or a thing, idea and anything" (S6), and "everlasting number" (S21). There was even a sophisticated answer from one student, who said "goes on forever and forever. Never stops. No one can say infinity is a certain number because it goes on and on. Non stop" (S20), giving the idea that infinity is a concept rather than a number. Infinity is a difficult idea with much research on it, and though these student conceptions are all operational they show a movement towards the concept of integers having no upper bound.

It is evident from the creative responses given above that some students are able to write examples of large numbers, compute with them and also give examples of situations that use large numbers. There was also some evidence of the use of exponential notation and many of the answers to 5a(ii) showed multiplicative thinking. Somewhat parallel to the historical development, S6 changed his scale of counting from 100 to 10000 when asked to count faster. S21 explained his method of counting faster as "It makes you think a lot for question 4. I wrote that pattern because instead of counting 1 2 3 ….you could just do it by just say in 1000, 100000, 1000000000…because it makes it much easier if you are counting with large numbers". S9's thinking was similar to S21 when she wrote "999+999000+999000000…and so on". S7's answer to question 4 showed structural understanding reflected in the general statement he made as an answer to how to count faster: "square it by itself (times it by the original number)". We suggest that S7, S9 and S21's answers tend to confirm Zazkis's (2001) theory that contemplation of large numbers encourages students towards a sense of structure. Also the emphasis on operations without actual computing (Hewitt, 1998), may help students, for example S7, towards abstraction. With regard to Q6 on infinity S18, S15, S13, S20, S6, S19 and S21 all demonstrated a well-developed operational sense of the concept at that level. It is interesting to note that S6's idea of infinity is not limited to numbers, and S20 does not see it as a number. As noted by Irwin and Burgham (1992), several students showed that they were thinking beyond their curriculum level in relevant aspects of the questions.

Some students commented on the relevance of some of the questions, and this may, perhaps have accounted for their non-participation. It seems that if questions 5a and 5b had had more contemporary value they might have elicited a more keen response from the students, and this is noted for future research. However, others were more motivated and there was enthusiastic participation from many of them. Still, it was not an easy exercise and this is reflected in their comments: "It's confusing. All that figuring out takes a lot of time" (S18); "This is quite hard, complicated" (S15); "Big numbers are hard. Numbers never end. Think outside of the box. You have to think about numbers that you wouldn't normally use" (S19). The fact that some questions were open ended also unsettled two students. S17 did not answer any of the questions and she wrote; "I'd rather work the questions out and there be a right answer. And work towards the correct answer". S20 said; "In my head a lot of these questions don't make any sense and is impossible to have a correct answer. If I can't get an exact answer instead of an estimate it confuses me and makes me unsure about the question". Others were more positive. "It makes me think outside of the box. Making me use different methods" (S4), "I like making hard examples of large number situations. I would like to find them out but it'll take a long time and maybe some of them won't have answers" (S2), and "It makes me think of the ways of getting bigger numbers or solving big numbers in easier ways" (S7).

Conclusion

The historical development in India of the present decimal number system provided the inspiration and impetus for this research, which has begun an ontological investigation of Year 9 students' ideas of large numbers, with the aim of progressing to learning place value through experience of exponentiation. The results suggest that an understanding and use of large numbers, and even of exponentiation and infinity, is accessible to students who have not been explicitly taught about these, as with our students. Furthermore, we hope that in future an explicit consideration of large numbers may help students to gain an awareness of number system structure without experiencing the historical limitations in European mathematics. It is hoped that sustained practice in naming and working with large numbers will lead to a development of further structural knowledge, with implications for understanding place value. We have not yet linked consideration of large numbers directly to the knowledge of positional notation, but intend to undertake such a study in the near future, and expect this will help formulate a framework for future research on students' understanding of the number system.

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