# The Introduction of Angles

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*This paper reviews some of the problems with the current frameworks for studying geometry curriculum at the primary level. This is done based on the experience of an ongoing project involving the development of an alternative geometry trajectory. The results presented indicate the possibility of introducing the angle concept much earlier than what is currently the case. A leading role for instruction to create a zone of proximal development is suggested.*

## Introduction

During in-service workshops, it is not very uncommon to find teachers who are not willing to accept that a square is also a rectangle. It is also common to find children and also adults who do not recognise a square on its vertex as a square and immediately call it again a square when it is put back on its side. This phenomenon existing in spite of the focus on teaching of shapes from kindergarten onwards raises questions about the efficacy of the current teaching practices in primary and middle school geometry.

For the last twenty years or so there has been considerable research interest in the teaching of traditional school geometry. A shift seems to have occurred internationally from the earlier focus on curriculum innovations inspired by the Topological Primacy Thesis of Piaget. In the current period, the level theory of geometrical thinking proposed by Dina van Hiele-Geldof and Pierre van Hiele has been very influential. This theory proposed in 1957 influenced the research and practice in Japan and the erstwhile U.S.S.R. Since the eighties their theory also had a major impact in the research in the United States (Whitman et al. 1997; Burger and Shaugnessy, 1986).

The van Hieles proposed their theory based on the classroom practices of Dina van Hiele-Geldof in her class of 12 year olds. Their theoretical framework was essentially in the phenomenological tradition and was influenced by Gestalt psychology (van Hiele-Geldof, D., 1984/1957, pp. 68-69). Dina van Hiele-Geldof used tilings in her class to orient her pupils towards recognising parallelism and similarity and the ability for deduction through five phases she termed as information, directed orientation, explicitation, free orientation and integration. The van Hiele theory proposed five levels in geometrical thinking (van Hiele, P.M., 1984, pp 249-250). Currently these levels are identified as (1) Visualization Level (2) Analysis Level (3) Ordering/Informal Deduction Level (4) Formal deduction Level and (5) Rigour Level applicable only at university stages.

What was significant in their theory was that the transition from one level to the next was not considered spontaneous and was strongly dependent on instruction. Each level was considered to have its own language and part of the reason for failure was seen in the fact that teachers in the middle school conducted their classes in the language of Level 2 while the students might be at Level 1 or even Level 0. They explained the failure of middle school geometry instruction by pointing out that “two people who reason at different levels cannot understand each other.” (van Hiele, P.M., 1984, p. 250).

The van Hiele levels were tested by many researchers through interviews with children and found to be valid (Usiskin and Senk, 1990). The van Hiele levels were modified by researchers who investigated it further and one major modification is that proposed by Clements and Battista that there is a prerecognitive level before Visual Base level in which children 'attend to a subset of a shape's visual characteristics and are unable to identify many common shapes or distinguish among figures in the same class.' (as cited in Clements et al. 1999, p. 193). The van Hiele levels have also been researched and put into practice in a major programme in South Africa (Human and Nel, 1997/87).

The van Hiele levels were developed in the context of the middle school curriculum and it identified the serious problems that emerge due to the long period of geometrical inactivity between the kindergarten level and the beginning of middle school. The activities to be conducted at the primary grades in order to support the transition of children from a Prerecognitive level to the level of Analysis through the Visualization level needs to be worked out so that they can participate in the informal deduction characteristic of middle school.

## Alternative Trajectory

In this paper I present the observations and insights from an on-going programme for the development of a primary mathematics curriculum in Delhi. The main purpose of this programme is to develop a sustainable community of practice. As a part of this experimental practice an alternative trajectory for geometry including traditional geometry is being tried out for the last one year, based on the experiences of the two years before that.

A working hypothesis leading the trajectory on traditional geometry has been the assumption that the lower van Hiele levels that have been posited could be also considered (at least partly) as a result of the nature of the existing form of instruction, and not an inevitable hierarchy of levels. The suggestion by Clements and Battista not only to include a Prerecognition level but also to rename the Visualization level as a Syncretic level, can be considered as an indication of the problems involved (as cited in Clements et al.1999, pp. 206 –209).

The hypothesis has been that the current phenomenon of children remaining at the Prerecognitive level as far as shapes are concerned could be influenced by the 'show and tell' approach in the teaching of geometry at the pre-school level. While at many schools concrete manipulatives are used in this process of 'show and tell', very often it is limited to showing the drawing of the shapes on paper including that of the three dimensional shapes. There have been experiences in which many older children have identified a triangle on its vertex as a cone. The alternative trajectory planned for actual manipulation of materials by the children including construction of the shapes.

The plan was to start from straight lines and angles before the 'introduction' of shapes. While children play with pattern blocks to make figures of their choice in free play at the pre-primary level (and later), there was no introduction of the shapes as such. Straight line and curved line were introduced at the pre-primary level. At Grade 1 closed figures and open figures made of straight lines/ curved lines were introduced. The introduction of angles followed this. It is only after the introduction of angles that formally triangles and 'quadris' were introduced as three sided and four sided closed figures. Due to the manner in which they are introduced there is no specific prototype that is introduced. Children make different kinds of triangles and quadris as a natural part of this activity. This is later followed by the introduction of rectangles and then squares. Given the limited space in this paper, I present the details of the introduction of only the angle activity and its assessment.

## The Didactics of Angles

Angles are normally introduced to children quite late at the primary level. The syllabus formed under the new National Curriculum Framework in India introduces this concept only at Grade 5 level when the children are 11-12 years old (NCERT, 2006, pp. 70-71). Yet at the same time if we consider the curriculum, children are expected to distinguish squares and rectangles from other quadrilaterals. The problems due to this have been already noted by researchers. In my own experience also I have come across students at high school who could not identify the angles in the shapes whose formulae for area and perimeter they were trying to learn.

Even when children are able to identify an angle, many of them have been known not to be able to compare the sizes of angles, indicating a very weak idea of angles. To begin with children tend to identify the size of the angle with the length of the arms of the angle and later also with the size of the opening at the base of the angle or the length of the arc marking the angle.

In the early research literature this led to a preference for the development of angle concepts based on turning rather than on the mutual inclination of two straight lines. It is in this paradigm that many activities were designed for children using the graphics-based computer programming language LOGO. But a review of such attempts of 'LOGO turtle geometry' has reported at best mixed results as far as the learning of angles is concerned (Clements, Battista and Sarama, 2001, pp. 7-8; Mitchelmore 1998, p. 266). In the Piagetian framework also angles are considered to be introduced later due to the need to co-ordinate the two parameters of length of arms and their linear separation (Piaget, 1960, p. 177).

In a series of important contributions, Mitchelmore and White seemed to have evolved towards a position giving primacy to the standard angle concept based on the relative inclination of two lines meeting at a point. To begin with, they distinguished between 14 physical angle contexts and based on interviews with children came to the conclusion that everyday concepts formed from these contexts cluster around corner, slope and turn which are based on the similarities of having two visible lines, a single sloping line and rotation. (Mitchelmore and White 2000, p. 232). They concluded that the standard angle concept develops in situations where both arms of the angle are visible. Mitchelmore and White followed up their interviews with children with the development of a sequence of lessons to teach angles. The lessons dealing with 2-line angles such as corners, scissors and body joints were taught in Grade 3 and the lessons related to 1-line angles such as doors, clock hand slopes were taught to Grade 4 (White and Mitchelmore, 2002, pp. 4-404). A recent doctoral dissertation in Norway experimented with using turn contexts to teach angle contexts to children and then finally resorted to a climbing context to attain success in the teaching of angles to 12 year olds (Fyhn, 2007, 2008).

## Angle Activity in the Classroom

My decision to introduce angles at Grade 1 level was decided by the intuitive sense of the children where I regularly teach and develop the curriculum. The angle classes as well as the geometry trajectory were developed in this school for first generation learners in Delhi, run by a social organisation Jodo Gyan working in activity-based education. This trajectory has been also shared with teachers of private schools and the resource persons of Jodo Gyan and currently more than 1000 children from pre-primary to class 5 are involved in this programme.

### *Angle Activity*

The angle activity starts with a discussion on corners. Children are asked how many corners there are in the classroom. Usually they begin with 8 and it quickly goes over 30- 40 and 100 as children start seeing corners everywhere in the room Then I take out two thin bamboo sticks and say,

*I:* Let us say I have two straight lines in my hand. Now can any of you come forward and make a corner with it?

That is always a moment of anxiety, but so far some child in the class has come forward to make a corner with the two sticks as shown below. This challenge is also an activity that usually gets the attention of all the children.

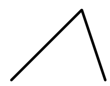

**Fig. 1.** Corner formation

Once a child does this, two straws and a flexible connector are taken out of the Geometry kit *Jodo* produced by Navnirmiti, with the words,

*I:* See, I have here with me something special with which we can easily make a corner.

Along with this statement two straws of the kit, connected with a flexible connector are held aloft and shown to the class.

*I:* And look, I have made the angle of the corner large, larger, larger… and now I have made it smaller, smaller and even smaller.

While saying this the angle made with the two straws held together with a flexible connector is widened and then brought together. This episode is the core of the activity and we usually have the rapt attention of almost the whole class of 30 children.

This pivotal activity is followed by invitation to the children to come forward to make another angle that is the same as the angle that I have made or those that are larger or smaller. To begin with, straws of the same length as the original one are given and later ones that are of different lengths. These are then usually displayed on the board, seriated according to the size of the angles.

During this process care is taken to see that the measurement is done accurately. When children superimpose one set of straws above the other to measure the angle, some of them spontaneously make the two vertices coincide, while others do not. Support is given to children during this process of measuring if needed. After the pivotal activity, this process of measuring is the crucial activity and can be considered to consolidate the understanding that children might have developed about angles.

It is felt that the two words in Hindi for corner (*kona*) and angle (*kon)* created a lot of problems. The thought of replacing the Hindi word *kon* with angle was quickly dropped and the activity continued with *kona* and *kon*. At a point, a (precocious) 5 year old was seen explaining to an older child *Dekho, kon or kona juduwa bhai he. Jidhar kona he udhar kon bhi he!* (Corner and Angle are twin brothers. Where there is a Corner there would also be an Angle!)

### *Right Angle*

In Grade 2 after consolidation of the idea of angles, it was followed with the introduction of right angles. This also took place through the posing of a challenge. With two thin bamboo sticks/straws held aloft, I asked

*I:* I have with me two straight lines and which one of you would like to try to make *four* angles with these two sticks?

In this case too there are usually volunteers and they easily crossed the sticks to make four angles. This was followed by asking, whether *all* the four angles could be made equal. That task also did not turn out to be difficult and the class agreed that the volunteer had got it right. During this process the teacher uses the words carefully and according to the language of the class slips in the word into the conversation of the class.

I: Have you got it right? Oh! See! Now we have four right angles and they are all equal to each other (*Kya abhi charo kon samaan he, baraabar he? Oh, dekho abhi hamare pas chaar sam kon he).*

This is followed by all the children in the class making the right angles they have made using straws as well trying to make right angles with their arms and other body parts.

The next day time is spent on identifying right angles in the room and children identify them all around including on the room cooler and on the shawl worn by the teacher.

Another activity to take the process further was checking out the geometry kit Rangometry (pattern blocks) to see whether they could identify any right angles within it. The right angles identified in the square are used to check the angles of other shapes. The angle trajectory could be done in 7-9 periods of 50 minutes each.

## Assessment

A first round of assessment of the activity has been done with the children who are studying in the non-graded class that can be broadly identified with Grades 1 and 2.

I assessed the children in one-one interaction or teaching interviews (Fuys, Geddes and Tischler, 1988, p. 12). We used the ray diagrams for post tests used by Fyhn for assessing children of Grade 6 in Norway (Fyhn, 2007, p. 30). The assessment with the ray diagrams on paper was preceded by asking the children to make angles larger than the angle that I had made using shorter straws. They were also asked to make smaller angles with longer straws. All the children were able to do this. During this process of checking, whether they had actually made it bigger or smaller as the case maybe, I worked with them to see to it that the vertices actually coincided during the comparison. There was also some conversation sometimes about the long *baaju* (arms) of the *kon* (angle) and so on. After this joint measurement activity the children were given the assessment sheet (Figure 2) to identify the biggest angle and the smallest angle.

All the children except a 10 year old who had joined the school two months earlier were able to identify the angles

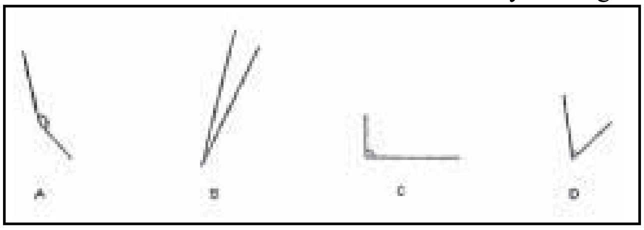

**Fig. 2.** Post test

| Unit topic | Activities |
|---|---|
| Angle | Challenge - Corner making with two sticks. Same angle/ bigger angle/ Smaller angle -1 period |
| | Making figures with straws and counting the angles and identifying biggest angles smallest angle -1 to 2 periods (Introducing Triangles and Quadris) |
| | Making figures of choice with straight lines on paper and identifying biggest angles / smallest angles -1 period |
| Right Angle | Challenge - making four equal angles with two straight lines -1 period |
| | Making right angles by children with straws- body parts -1 period |
| | Identifying right angles in the room - 1 period |
| | Identifying right angles in pattern blocks - 1 period |
| | Using the square to measure angles in other objects - 1to 2 periods |

**Table 1.** Angle Activities in Grades 1 and 2

correctly. This boy was able to make the larger and smaller angles with the straws during the interview but he could not go further to explore right angles. The age of the children ranged from 5 years and 9 months to 14 years. The age composition and student response is given in Table 2.

This, for example, compares favourably with the results of the study reported by Fyhn. In her study out of the 13

| Age | No of children | Identification of biggest and smallest angle |
|---|---|---|
| 6 years | 3 | Both correct |
| 7 years | 4 | Both correct |
| 8 years | 3 | Both correct |
| 9-10 years | 1 (new student) | Both wrong (focused on length of arms) |
| 12 - 14 years | 3 | Both correct |

**Table 2.** Assessment of Angle Concept

twelve year students of Grade 6 who undertook the same post-test, 1 was not able to identify the largest and the smallest angle correctly and 4 were not able to identify the smallest angle correctly (Fyhn, 2008, p. 23). In our case all the 6 years were also able to identify C as a right angle.

These results indicate that the teaching of angles lies within the zone of proximal development of the children of Grade 1, corroborating the impression that was created in the class. This is also supported by the results of the interview schedules from the children who had just entered Grade 2 in a private school where I took a session for about 40 minutes to introduce angles. In the assessment that followed two days later only 4 out of the 29 children identified the long armed angle B as the largest angle. 14 or 43% of the children identified the slightly obtuse angle A as the largest angle and another 43% identified the right angle C as the largest angle. The age of the children ranged between 6 years 7 months to 7 years 8 months. If a single session can change children's perception of focusing on the arms, then there seems to be support for the argument that the idea of angles lies within the zone of proximal development of 6 year old children.

## Zone of Proximal Development

The assessment and the Grade at which angles need to be introduced bring us necessarily into some theoretical discussions on the meaning of the zone of proximal development. It would appear that there are very different interpretations of this term that Vygotsky introduced. Apart from making a correct assessment of the original term it is also necessary to consider how it needs to be further taken forward. The concept of the zone of proximal development is intimately connected with the relationship between the everyday concepts and activities which get introduced from outside with *conscious awareness and volition*. For Vygotsky, the zone of proximal development is being created on the one hand by a certain level of development of the everyday concepts and on the other hand by instruction which develops a conscious conceptual structure.

> The strength of the everyday concept lies in spontaneous, situationally meaningful, concrete applications, that is, in the sphere of experience and the empirical. The development of the scientific concepts begins in the domain of conscious awareness and volition. It grows downward into the domain of the concrete, into the domain of personal experience (Vygotsky, 1987/1934, p. 220).

This also brings to the fore the point that has been repeatedly stressed by Freudenthal that the formation of mental objects needs to precede the formation of formal concepts and that 'in no part of mathematics do mental objects serve so long before, or even without, concept formation as in geometry' (Freudenthal, 1983, p. 226). The everyday concepts of Vygotsky and the mental objects of Freudenthal could be considered to belong to the same genre. We could argue that instruction could play a role in the further development of these mental objects/ everyday concepts also.

Even as there are similarities between Piagetian concepts of empirical and reflective abstraction there are also fundamental differences. The path of introduction of the scientific concepts and their trajectories of growth are different. Here it is not a question of cognitive conflict but of collaboration and joint activity. Leontyev gives the example of a child learning to drink from a cup to show the nature of the interaction which takes place between the child and the adult and about how a child *appropriates* the cultural traditions of her world around. He says that

> The fundamental difference between the processes of adaptation… and the processes of appropriation and mastering is that the process of biological adaptation is one of change of the organism's species characteristics …whereas the process of appropriation or mastering is one that results in the individual's reproduction of historically formed human capacities and functions (Leontyev, 1981, p. 296).

In fact when we are discussing the learning of angle concepts we are discussing an issue of appropriation. Vygotsky suggests that 'collaboration and imitation is the source of all the specifically human characteristics of consciousness that develop in the child" (op cit., p. 210). He also took the position that instruction and development do not coincide and that "instruction is useful only when it moves ahead of development." (p. 212) "We assist each child through demonstration, leading questions and by introducing the initial elements of the task's solution." (p. 209).

With this brief introduction about the role of instruction we can consider the question of appropriate Grade to introduce angles. The study by Mitchelmore and White has showed that "many children form a standard angle concept as early as Grade 2". In their interviews they used different type of tasks and in a large number of the tasks the children of Grade 2 showed very high ability to show their existing concept of angle. Thus for example, 87% of the Grade 2 children succeeded in the size matching task in the cases of scissors and fan and 77% in the case of door (Mitchelmore and White, 2000, p. 225). Yet when they introduced their lesson for 2-line angle concept they introduce it in Grade 3. This would appear to be a case of instruction following development rather than leading it. The encouraging results which we have obtained with the introduction of angles to children of Grade1 and 2 would indicate that it lies within their zone of proximal development.

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