Problem Solving In Mathematics: A Tool for Cognitive Development

Preety N. Tripathi

State University of New York, Oswego, USA

How can problem solving be used as a tool for cognitive development? Can problem solving be used to effect a change in learners' attitudes and beliefs about mathematics so that they come to view mathematics as a discipline founded on reasoning? What are some strategies that instructors may use in the process? To seek answers to these questions, I conducted a study with a group of prospective elementary school teachers. In this article, I describe briefly my attempts to answer the above questions.

Introduction

Historically, learning mathematics and teaching it to all students at the school stage has been motivated by the belief that a study of mathematics helps students to learn to reason and apply such reasoning to everyday problems. It is believed that learning mathematics leads to learners' cognitive development. Thus, one of the important questions that all mathematics educators must constantly ask themselves is: Does the mathematics that we teach (and that our students learn) lead to an enhancement of students' cognitive abilities?

This leads us to clarify what we mean by the understanding of mathematics that we seek to develop in our students. The deeper understanding that we are looking for must enable students to look at and understand a new situation, delve into the repertoire of mathematical knowledge that they have in terms of concepts, processes, and ideas and adapt or modify those ideas so as to apply them towards resolving a new problem situation. Such understanding calls for building deep connections between concepts, a variety of lenses and representations with which to view the concepts, and flexibility that allows one to sufficiently modify concepts so as to apply them to a new situation. It requires students to develop a rich network of ideas that one may draw from when faced with a novel situation. In this process, students develop habits of the mind that enable them to analyze other situations that they may encounter in life, mathematical or otherwise. This critical blend of processes is what mathematics educators refer to as problem-solving. It is this kind of cognitive development that most modern societies would like their citizens to develop.

This article addresses the question: How can students be taught problem-solving in the mathematics classroom? More importantly, can students' beliefs about mathematics teaching and learning be influenced via the teaching of a course in problem-solving? What strategies might promote this belief?

Review of the Literature

Problem solving, as used in mathematics education literature, refers to the process wherein students encounter a problem – a question for which they have no immediately apparent resolution, nor an algorithm that they can directly apply to get an answer (Schoenfeld, 1992). They must then read the problem carefully, analyze it for whatever information it has, and examine their own mathematical knowledge to see if they can come up with a strategy that will help them find a solution. The process forces the reorganization of existing ideas and the emergence of new ones as students work on problems with the help of a teacher who acts as a facilitator by asking questions that help students to review their knowledge and construct new connections. As the new knowledge is embedded into existing cognitive frameworks, the result is an enrichment of the network of ideas through understanding. The simplified process described above was first summarized in Polya's path-breaking book (1957), and has since then inspired much research.

It is the worthwhile search for mathematical growth that has researchers looking for ways by which one may use problem solving as a teaching tool. The *Principles and Standards for School Mathematics* (National Council for Teachers of Mathematics, 2000) describes problem solving based teaching as using "interesting and well-selected problems to launch mathematical lessons and engage students. In this way, new ideas, techniques and mathematical relationships emerge and become the focus of discussion. Good problems can inspire the exploration of important mathematical ideas, nurture persistence, and reinforce the need to understand and use various strategies, mathematical properties, and relationships" (p. 182). This succinct statement encompasses about two decades of research and reflection on the entire gamut of issues related to problem solving in mathematics education. Even so, researchers continue to grapple with the issue of teaching via problem solving.

Research on problem solving emphasizes the role of the teacher in developing students' reasoning skills. As Weber (2008) avers, "To lead students to develop accurate criteria for what constitutes a good argument, the teacher must have a solid understanding of these criteria" (p. 432). Wheatley (1992) proposed that problem-centered learning is a teaching method that encourages student reflection, and presented examples demonstrating that encouraging reflection results in improved learning. It is these bodies of research that led me to believe that the audience of pre-service teachers was the perfect audience for a course in problem solving.

Literature on problem solving in mathematics has discussed extensively the need to teach students to reason mathematically. This train of thought led to an emergent theme in mathematics education in the mid-80's wherein researchers propounded that teaching mathematics via problem solving was the correct way to foster students' problem solving and hence reasoning skills. Schroeder and Lester (1989) contended that in mathematics, problem solving was not a content strand but a pedagogical stance. To elaborate, the researchers proposed that in teaching any mathematics class at any level, students be exposed to a variety of problem solving tasks that require them to collate and analyze previous knowledge and yet offer a challenge. Problem solving was thus seen as a means of developing students' reasoning skills. The researchers were influenced by the classical work of Polya (1981) and Dewey (1933). Much work was also done in defining and identifying good problem solving tasks for learners as well as modes to implement such teaching via small group cooperative problem solving, expository writing, problem posing, etc. (e.g., Lester and Charles, 2003).

Teaching mathematics via problem solving seems like an attractive proposition, but while examining the seminal work done in this field, researchers driven by constructivist frameworks were forced to step back. As they developed linkages between theoretical research and practice in the field, they were forced to examine the question: How does one implement the process of teaching mathematics via problem solving? This question was related to the deeper question of changing the attitudes and beliefs of students who viewed mathematics as a bunch of definitions and algorithms that exist in isolation. Researchers suggest that the problem lies with students' classroom experiences wherein students find little scope or motivation for them to learn how to reason. Scholars argue that it is not appropriate to merely teach students how to reason. What is important is to build a case for students to learn to reason. That is, before we can teach students to reason, we must persuade them to feel the need to reason. Current scholarly thinking in problem solving has thus focused on the need to change students' attitudes and beliefs about mathematics.

In this context, Selden and Selden (1995) aver that students' early mathematical years are extremely important because it is in these years that students' attitudes begin to form. They believe that right from the early elementary stages children should be encouraged to reason through their mathematical activities. According to them, "Both weak validation skills and viewing proofs as ritualistic, and unrelated to common sense reasoning, may be partially traceable to the absence of arguments, *especially student-produced arguments*, in school mathematics" (p. 141, emphasis mine). But, to inculcate a culture where students learn to reason through all their activities, *teachers* must appreciate the importance of such reasoning.

Clearly then there is a strong need to focus on preparation programs of elementary school teachers in order to incorporate a culture of reasoning. We should instill in teachers the attitude that mathematics is about reasoning rather than rote memorization. Problem solving courses are an important linkage in developing such attitudes. Structures within the course format must encourage students to offer both critique and explanations, so that students do not just have to reason but actually believe that such reasoning is an intrinsic aspect of mathematics. One of the structures that foster the development of students' reasoning skills is small group cooperative work. A well-developed strand of research in this area helped me to use strategies that enable work in small groups. Researchers have documented the importance of selecting suitable tasks and facilitating peer interaction in attaining cognitive development during problem solving activities. This research (for example, Hurme and Jarvela, 2005) was the basis of my work in selecting appropriate tasks and determining my own role during classroom activities.

At the cognitive level, my goals were aligned to Vygotsky's four major criteria that distinguish between elementary and higher mental functions: (1) the shift of control from environment to the individual; (2) the emergence of conscious realization of mental processes; (3) the social origins and social nature of higher mental functions; and (4) the use of signs to mediate higher functions (Wertsch, 1985, p. 25). Each of these criteria can be clearly delineated in the process of mathematical problem solving and I aimed to help my students achieve these goals. Was I successful in achieving these goals? Would my students be able to transcend the barrier between elementary and higher mental functions as outlined by Vygotsky? These are the questions that I attempted to answer in my study.

The literature review described above points to the following questions: How does one help students to learn to believe that it is essential that pre-service elementary school teachers learn to reason mathematically in all the situations encountered in school mathematics? The last question has much to do with teachers' attitudes and beliefs structures about mathematical knowledge, and thus by nature is more difficult to answer. However, a solution to this problem is indispensable because it is directly related to students' cognitive development. I amplify this statement below.

Details of the Study

The class of prospective elementary school teachers that I taught was located in the mathematics department of a state university in central New York. The course was a required course for students who declared a concentrate in mathematics. There were 28 students in the class. I divided the 28 students into small groups of four students each. The students would sit in their assigned groups on each class day, and after a brief introduction to the problem, start working on the problems. The textbook contained problems whose content was concepts drawn from the elementary school mathematics curriculum presented in a problematized context. The book contained genuine problems for the students as they could not be solved by any readily available algorithm that students already knew.

My role in the class was that of a facilitator. I went to each of the seven groups and listened to their discussion. I would often interject with a question to push for reasoning, justifications or explanations; I did not offer any solutions to the problem. We attempted to maintain a spirit of enquiry and critical reasoning. My questions would ask students if they could organize their information differently, if they could think of a similar problem that they had done before, or a concept that seemed related. I would also initiate and moderate class discussions when students from different groups would share their work on the problem. The small group interactions and the class discussions were meant to help students to build connections, choose among different means of organizing data and solving the problem, and to extend their knowledge of concepts.

Data was collected and analyzed qualitatively. It was collected via notes before, during, and at the end of the class, from students' work during class, and on homework and quizzes, and through reflective notes that I wrote at the end of the class.

Data from the Study

Episode 1

I describe below students' first attempt to solve a problem in the class. The problem is a classical game often used in problem solving classes. In using this as the introductory problem in the class, I had two goals: to initiate students' work in small groups, and to introduce them to arguably the most basic strategy in problem solving – creating and organizing data so as to yield useful information. Students worked on problem 1 using concrete tiles. Each group of four had two teams that played each other in pairs.

Problem 1: A pile of 20 tiles is placed in front of you. Each team, in turn, must take either 1 or 2 tiles from the pile. The team that gets the last tile will lose the game. Take turns to go first in the game. After playing a few rounds, can you predict who the winning team will be? What would be your strategy to win the game?

Students were prompted to play the game a few times, alternating first turns, and keeping data on who won each game. Gary showed me his lists. For each game he had made a list that showed how many tiles had been removed. His group had figured that the team whose turn it was when there were four tiles on the table would lose the game. Several other groups said that they had reached that conclusion as well. I responded by saying that this looked like a good first step. I also challenged them to see if they could predict the winner earlier in the game.

Gary: Oh, you mean like, when, right at the start?

I: Yes, that's what we would call "a winning strategy."

Gary: Can we know that by looking at our lists?

I: Perhaps. You might also play the game a few more times so that you have more data.

Several groups asked me if I could help them because they were lost. I then asked them, "Which is the next higher number that would ensure a winner?" This question had them looking at their lists to find a definite number. From here on, students themselves looked for the next higher number and finally predicted that for 20 tiles, the first player must take one tile to win. Problem 1 was followed by a series of questions to help clarify students' thinking. In different questions, the starting number was changed, the number of tiles being removed was altered, and tiles were added instead of being subtracted. For each of these situations, students worked with the tiles in their groups. At the end of the class, most groups had at least two or three more similar questions to answer. I asked them to work on these questions as homework.

Analysis

This first problem was meant to be an "ice breaker" so that students who had not worked in cooperative groups before could get a feel for group work. As one of the primary aims of the class was to have students develop skill in problem solving, I also focused on introducing students to several strategies – using concrete manipulatives, creating organized lists, looking for patterns, working backwards and solving a simpler problem. After the first question, the objective was to help students clarify and extend their thinking. For different questions, students modified their thinking a little as they struggled to re-organize their framework. The homework aimed at helping them in this process.

Different groups created different formats of data organization before they found one that worked for them. This task in the form of a game naturally motivated the need to reason (in order to win). As part of my research question about changing students' attitudes and beliefs about mathematics, we also addressed the question of "where is the mathematics in the game?" and students came up with different answers, ranging from "because there are numbers involved" to "the way we did it, it involved logical thinking and that is mathematics."

Episode 2

The second episode that I describe here happened three or four weeks later. At this time, students were a little more used to the class and to small group cooperative work. The background for this problem was familiar to all students from school. The problem asked them to examine this old knowledge, analyze it and develop mathematical reasoning to show why it worked.

Problem 2: Justify the divisibility tests for the numbers 2, 3, 4, 5, 7, and 11.

Students tried to justify the test using base blocks, pictures or expanded notation of numbers. During the ensuing class discussions, students came to the blackboard to show their justification for each divisibility test, and even pointed out the advantages of the method that they had chosen. As students shared a variety of strategies, one student remarked, "we choose a way that we like, but sometimes other methods may work better."

Analysis

This problem was chosen specifically to focus on the idea of justification in mathematics. The discussion and analysis during the problem was directed at helping students to make conscious choices among different available tools to support their work. Besides, as learners suggested ways to improve upon the displayed pictures, they were developing their communication and reasoning skills. The variety of strategies that the students chose showed signs of independent thinking. My own reflective notes at the end of the day of this class showed that this was perhaps the first time that students made choices and conscious decisions about the use of tools and strategies. Students also gave reasoned responses to justify their own decisions or why they preferred one strategy over another.

The process of development was extremely gradual. Each time we began a unit, I would discuss the ideas that they should expect to see in the unit. I also needed to clarify basic definitions of concepts as and when there was a need as students seemed to struggle with these ideas. The process also brought up some basic ideas about the structure of mathematics: What are definitions good for? What is a theorem? What role do counterexamples play in building the discipline? I believe that these clarifications helped learners to get a better understanding of mathematics. For the students, the best part of the class was working in small groups. It gave them the opportunity to test out their ideas in a safe setting, and discuss and modify them as they progressed with each other's help. Slowly students started to participate more in the group work.

As the class progressed, I saw some definite gains in students' problem solving abilities as students talked about using a smaller number, or working backwards, or looking for a pattern. Students became better at organizing information and choosing among different representations to show not only their answer but also their reasoning. I also found a perceptible increase in the number of "how" and "why" questions being asked in class. Other changes were more subtle to perceive. Students came to realize that they would consistently see problems in the course, even on exams and quizzes. This, I believe, was a big change for students who had steadfastly held on to the notion that, "at least in an exam, you can't expect us to think on the spot." More importantly, they had started to believe that they did have the tools, skills and knowledge to actually solve many of these problems. With some students, I saw an increase in interest in problem solving, as they came to me to ask me to suggest books or websites that they could read outside the course. Clearly, these were positive signs. But had I succeeded in changing some of their beliefs about mathematics? I attempt an answer to this question in the following section.

Discussion

One semester is too short a period of time to effect a change in the attitudes of any group of people and I was well aware of this constraint when I started. Yet, I believed that even if the process could be initiated during this time it would be the beginning of a new approach to teaching and learning mathematics. The success of my course rested on the building of a classroom environment wherein asking the question "why" was a matter of course. I hoped to develop a classroom culture where students felt free to make conjectures, and then verify those conjectures "physically" (mostly, by looking at specific cases of the abstract case under discussion). It was then that they would try and prove their conjectures. I did not always expect this proof to be formal or rigorous. The proof could be informal and yet had to have a certain degree of rigor – thus, students could provide visual proofs (as they did when working with base blocks for the divisibility tests), they could show existence of a concept by actually building up the concept (as they did for some of the geometric constructions), and could develop new algorithms of their own as long as they could explain why the algorithm worked. I found that students did start to ask "good" questions - they were asking their peers to defend their solutions, providing situations where a given solution may not work or providing a different solution. They would compare different kinds of solutions and representations and try to find linkages between them. Thus, I believe that we did make a start in working towards satisfying the four criteria described by Vygotsky for higher mental functions. I discuss this statement below.

I believed that the two most important factors in influencing students' beliefs about mathematics are the culture in the classroom and the nature of the mathematical tasks that students work on. The groups allowed students the opportunity to communicate in a spirit of enquiry. Thus, they could develop and verify conjectures through their discussions. The most important function of the group was to provide a safe medium wherein students could challenge themselves and others through questions and try to find answers. This approach is in line with researchers' contention that mathematics is developed through socially-mediated cognitive experiences. The problems themselves were suited to such activities, and in follow-up questions some conditions were changed so that students could reason back through their solutions, and construct arguments in different contexts.

In the process of solving the problems, often students would re-organize their information, re-construct arguments and look for different contexts to verify their ideas. For example, in problem 1 listed above, students worked with different tools – concrete (tiles), visual representations (drawings), or just numbers. They tried to work backwards or to simplify the numbers, or change conditions in order to find answers to the problem. As they attempted to answer this question and the follow up problems on the task, they would begin the process of reflection by asking themselves, "So what am I doing here?" "Does this help?" "What if I re-arrange it like this?" Thus, I found that students were beginning to take control and to self-regulate their solutions. Gradually, they began to make choices about their activities, "should I try this with tiles?" "Why don't you draw what you are saying?" were some of the questions I heard as the semester progressed. I believe that these questions reflected metacognitive activities where students became aware of their own mental processes, and attempted to negotiate through them. Secondly, I also found that students started to make choices about which tools they should use in order to best solve their problem. Often, they would use different tools at different stages of the problem. For example, they would begin by using concrete tools for small numbers and make organized lists as the concrete situation became cumbersome. Once they could find some discernible pattern in the data they tried to use a more abstract tool such as an equation. Thus, I found some evidence of the use of tools and symbols that the students used to mediate activities.

The third criterion of socially mediated arguments was a by-product of small group work. As students became comfortable with their groups, they insisted that all members explain and communicate their work. Students would reason verbally, or through representations such as pictures, charts or symbols. I encouraged them to come to the board though few of them felt comfortable with this strategy. The small group medium provided a channel through which they could scaffold their reasoning processes. Mostly, they presented informal justifications, but often while working to develop such proofs their work would become more rigorous and formal.

In all, I feel that there was some success in steering students towards the belief that mathematics is about enquiry and cognitive challenges, and that one grows by taking on these challenges and trying to find a reasonable solution. Students' gains in acquiring facets of Vygotsky's four criteria of higher mental functions suggested that I had had some success in meeting my goals for my course. For these students this was a good beginning, and I believe that they needed to build on the skills that they had acquired in the course.

Conclusion

In mathematics education research, problem solving has been closely linked to cognitive development for at least the last twenty five years. Research in this area has stemmed from a re-conceptualization of mathematical thinking. This alternative epistemology of mathematical knowledge has been less focused on conceptions of domain knowledge. Instead, it tends to emphasize metacognition, critical thinking and mathematical practices as the critical aspects of mathematical thinking. To translate this theoretical perspective into the classroom, we need to bring about a fairly revolutionary change in our attitudes and belief structures about mathematics. It is critical that we begin this change with educating our elementary school teachers who are the agents of change in school classrooms where they influence the attitudes of very young learners. My work in this project suggested to me that we do have a fighting chance of success. At the stage where we meet prospective teachers, we can influence how they view mathematics as a whole and consequentially, bring mathematical reasoning to the forefront of our discussions in our content and pedagogy classes. But, in order to do this, we need to turn the lens on ourselves so as to examine our own pedagogies in the classroom. I believe that my study offers some insight into teaching such a course. The entire process is difficult and time-consuming, but given what is at stake, I believe it is well worth the effort.

References

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educational process*. Boston, MA: D.C. Heath and Co.

Hurme, T., & Jarvela, S. (2005). Students' Activity in Computer-Supported Collaborative Problem Solving in Mathematics. *International Journal of Computers for Mathematical Learning*, 10(1), 49-73.

Lester, F., & Charles, R. I. (Eds.) (2003). *Teaching mathematics through problem solving*. Reston, VA: National Council for Teachers of Mathematics.

National Council for Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Polya, G. (1945; 2nd edition, 1957). *How to solve it*. Princeton, NJ: Princeton University Press.

Polya, G. (1981). *Mathematical discovery: On understanding, learning and teaching problem solving* (Combined ed.). New York: John Wiley and Sons.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, 29(2), 123-151.

Schoenfeld, A.H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Schroeder, T.L., & Lester, F.K. (1989). Developing understanding in mathematics via problem solving. In P.R. Trafton (Ed.), *New directions for Elementary School Mathematics, 1989 Yearbook of the NCTM* (pp. 31-42). Reston, VA: NCTM.

Wertsch, J.V. (1985). *Vygotsky and the social formation of the mind*. Cambridge, MA: Harvard University Press.

Weber, K. (2008). Mathematicians' validation of proofs. *Journal for Research in Mathematics Education*, 39(4), 431-459.

Wheatley, G. H. (1992). The role of reflection in mathematics. *Educational Studies in Mathematics*. 23(5), 529-41.