Achilles and the Tortoise Paradox – Finding the Zone of Proximal Development in Understanding Limit of a Sequence

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We discuss here the outcome of a Teaching Experiment conducted in the classes of Freshman calculus in CUNY colleges of the Bronx, NYC, which was supported by the NSF-ROLE grant #0126141, Introducing Indivisibles into Calculus Instruction. One of the main issues investigated was student understanding of the concept of the limit of sequences. The teaching used the guided inquiry (or guided discovery) method. Here we report students' work with the Achilles and the Tortoise paradox which involves an infinite converging sequence. Attempts to identify the zone of proximal development with respect to this problem, together with appropriate modifications to the presentation of the paradox in the form of assignment problems are described.

Introduction

The discussion presented is the outcome of a Teaching Experiment in which we investigated student understanding of the concept of the limit of sequences. The course was led with the help of the 'guided inquiry' (or guided discovery) method, which relies on the formulation of challenging open tasks while leaving students enough cognitive space to make substantial discoveries and insights by themselves. To a large degree the art of facilitation of student discoveries relies on teachers, familiarity with cognitive steps and cognitive distance needed to reach them.

The *cognitive space*, according to Benking (2008) is the individual perceptive capacity, resulting in a unique characterization of ideas. The dimensions of the cognitive space depend on information, training and person's awareness. In mathematics the cognitive space is the mental inner environment of an individual within which mathematical concepts are formulated and organized through the formation of interconnections between them. The *cognitive step* is a single step (a single connection or a grasp) needed to reach an element of understanding. *Conceptual distance* is a sequence of such steps joining two different levels of understanding. The work of the teacher in the classroom of guided discovery requires a difficult skill "to pose questions which are sensitive to the bandwidth of competence within which each individual can navigate. The point is to map Zone of Proximal Development (ZPD)." (Brown, 1992)

The question of cognitive distance between different aspects of a problem is well understood with the help of the theoretical framework of ZPD of Vygotsky (Vygotsky, 1987). Zone of Proximal Development is the conceptual distance between the level of student understanding of the concept by himself/herself, and the level of understanding of the same that can be reached by the student with the help of facilitation by a mentor or an instructor. In the context of classroom instruction, the outer levels of ZPD are determined by the "spontaneous concepts" of students and "scientific concepts" of the instructor, and the mastery and understanding of the concepts takes place in the zone between the two. The art of the teacher-researcher lies in formulating such questions during the process of classroom facilitation, which allow the student to traverse that distance as much as possible on his/her own.

The Teaching-Research questions for the Teaching Experiment were:

1. What is the nature, extent and scope of ZPD related to the concepts of converging sequence and its limit amongst students of Freshman Calculus, in the context of the Zeno's *Achilles and the Tortoise* Paradox?

2. What could be the optimal path of hints for students to traverse their full ZPD?

The discussion of the case study is motivated by the teaching challenges of the guided inquiry method in facilitating student thinking. It's important to distinguish didactic difference between the open discovery methods when we want student to open his/her imagination to all the possible routes of thinking and its outcomes, and the guided discovery method when the teacher wants to facilitate student thinking to reach a definite, somewhat pre-established point in his discoveries. The goal of the teaching episodes described below, is for student to recognize (discover) the converging sequence in the movements of Achilles and Tortoise and apply correctly the definition of the limit; the goal is not, necessarily, the solution of the Zeno paradox.

Organization and methodology of the Teaching Experiment

The Teaching Experiment is a part of a larger study investigating student understanding of the limit of sequences, which is one of the first topics in Calculus followed by investigations of the limit of functions. The understanding of the limit of sequences is particularly important for understanding the Riemann construction of the area under an irregular curve. The data collected consisted of four different writing assignments, which were sequentially refined depending on student responses. The time span for the assignments was the first 6 weeks of the semester which are usually devoted to the limits and graphing of functions. The material needed for the successful response was worked upon during this time through the episodes of the guided discovery method. Students of the class started the study with no knowledge of sequences and their limits, with very weak algebra skills. The sequence of writing assignments below shows 3 Teaching Research cycles through which such an effect was reached revealing at the same time the didactic difficulties of the approach as well as the routes to student success. The assignments were given during the first 6 weeks of the semester and 13 students of the class had approximately a week per assignment to complete it. A decisive majority of students did not ultimately make the required connection; apart from the student discussed, there was only one other who understood the diminishing distance connection between Achilles and Tortoise, but due to an arithmetical error did not obtain non-trivial converging sequence.

The concept of the limit is a difficult concept where much of the attention has been focused. In the context of this article, the Teaching Experiment had already evolved through several cycles in which the formal definition of the limit, prone to student misinterpretation and hence misconceptions, had been replaced by the following geometric definitions, Versions 1 and 2, which accomplished the scaffolding needed for the formal definition of the limit.

The statements of each of the definitions mentioned above are stated in the instructional material as follows:

A sequence (a_n) has a limit L means:

(1) *Geometric Definition version 1*: For any two equally spaced horizontal lines with the point (0, L) between them, there are at most finitely many points outside the two horizontal lines.

(2) *Geometric definition version 2*: Whenever we draw two parallel lines around the limit point (0,L), we can always draw a vertical line such that the terms of the sequence of points (n, a_n) to its right are in-between the lines.

(3) *Analytic Definition*: For any $\varepsilon > 0$, there is a number N, such that $L - \varepsilon < a_n < L + \varepsilon$, for all $n > N$.

It is in this context that the paradox of Achilles and the Tortoise was used in the writing assignments to elucidate the nature of students' existing misinterpretations and to provide the scaffolding deemed necessary based on student work.

Teaching Experiment

Example: *Investigations of the Zone of Proximal Development of students in Freshman Calculus – Achilles and Tortoise (A&T) paradox as a converging sequence.*

The aim of the series of written assignments was to facilitate student recognition of the converging sequence in the Achilles and Tortoise paradox, and to apply the definition of the limit of such a sequence to calculate the limit. Understanding of the limit as a resolution of the paradox was desirable. Each cycle of assignments is refined by the results obtained from previous answers. The first assignment, in the Spring of 2004, served as the diagnostic test for the Fall semester.

Assignment 1 (Spring 2004)

There is a race between Achilles, the fastest of Greek warriors and the Tortoise. Achilles gives Tortoise a head start and is running after him. However, in order to pass the tortoise, Achilles has to arrive at the point where tortoise was before, from where however, tortoise already departed. Hence, having gotten to that point, to pass the tortoise, although the distance is shorter, Achilles still has to get to where tortoise is at present. However, by the time Achilles will get there, Tortoise will have departed again. Since this situation can continue with no end, Achilles can never overtake the tortoise.

Write a 1 page essay discussing the Achilles and Tortoise paradox. Make sure you include the following:

1. State where is the contradiction in the story. A contradiction has at least two statements which imply opposite conclusions.
2. Which of the two statements would you agree with?

The set of student responses received contained both views with quite a few who agreed with Zeno, and their arguments show quite a few misconceptions (student excerpts S1 – S4 below) that underline their reasoning. The misconceptions could be quite easily eliminated, teacher-researchers thought, with a better formulated assignment.

S1: "One is that you are not able to do an infinite number of tasks in the finite time. This relates to Achilles because he has an infinite number of finite steps to catch up, before he can catch up with the Tortoise."

S2: "It is obvious that if the distances get shorter each time, its only the matter of time, however long, till the Tortoise is overtaken."

S3: "I see that the Achilles will never catch up with the Tortoise because even though the Tortoise is slower, he is not stationed at the same location, which makes the story continue with no end."

S4: "It is almost impossible for Achilles to get to the point where the Tortoise is and then pass him, because according to Zeno, Tortoise is stationary. In addition if Achilles is to pass the Tortoise would have to be stationary."

It is interesting to note that the ways of understanding the A&T difficulties by students touch upon fundamental issues. S2 talks about the distance and her intuition tells her that there should be no problem summing up the decreasing sequences of their lengths. On the other, S3 points to the changing positions of the tortoise and on the basis of this intuition, suggests the impossibility of passing. Most striking is S1 who, similar to Grunbaum (2002), sees the paradox not in the summation of the infinite number of distances or time intervals, but in the need to perform the infinite number of actions in finite time. This, very profound point of view moves the A&T discussion beyond mathematics, indicating that the mathematical understanding of the situation using the concept of the limit, as for the first time done by St.Gregory in 17th century (Cajori, 1915) solves only the mathematical aspect of the paradox, and not its more intrinsic nature. Here, the teacher-researchers were interested in finding the best sequence of topics of the essays which would allow students to utilize the definition of the limit of a sequence for their understanding of the situation. In the words of ZPD, in this teaching experiment, its upper level was given in terms of the "scientific concept" of the limit and the investigations concerned the proper scaffolding to raise the lower level of "spontaneous, intuitive concepts" of students. The design of the themes of essays needed to be such that it would allow the integration of both spontaneous and scientific concepts into a proper understanding of the paradox.

Assignment 2 (Fall 2004)

In the fifth century BC the Greek philosopher Zeno of Elea posed 4 problems, now known as Zeno paradoxes that were intended to challenge some of the ideas concerning space and time held in his day. Zeno's second paradox concerns a race between the Greek hero Achilles and a Tortoise that has been given a head start. Zeno argued, as follows below, that Achilles could never pass the Tortoise. Given that Achilles is the fastest Greek hero, Tortoise is one of the slowest animals, the speed difference between is very big, so that the conclusion that Achilles can never pass the Tortoise is contradictory with everyday experience.

Zeno argues as follows: Suppose that Achilles starts at position a_1 and the Tortoise starts with the position t_1 having a head start in terms of the distance. When Achilles reaches the point $a_2 = t_1$, the Tortoise is already further away at position t_2. When Achilles reaches the point $a_3 = t_2$, the Tortoise is at position t_3. This process continues without end, and Tortoise is always ahead of Achilles. The conclusion is that Achilles can never overtake the Tortoise.

Achilles	a_1	a_2	a_3	a_4	a_5	...	...	....
Tortoise		t_1	t_2	t_3	t_4	...	...	....

Write a one page long essay proposing the resolution of the Achilles and Tortoise paradox.

A paradox is resolved if the contradiction between the conclusion as stated and the everyday experience has been eliminated.

In your essay,

1. State the paradox as you understand it. Make sure your interpretation of the paradox agrees with all facts in the story.
2. Propose the resolution to the paradox based on your careful rethinking of the process in which the paradox arises.

Note the introduction of algebraic symbols in the assignment attempting to symbolize the approach to the paradox. We are investigating here levels of algebraic comprehension of the text and student ability to repond to it.

The description of the paradox is more detailed and procedural, although the attached drawing indicates interpretation of the procedure resulting in the shortening of each successive distance. Below is one of the best student's response:

Achilles a_1 ——— a_2 ——— a_3 ——— a_4 ——— a_5 ——— a_6

5miles *5miles* *5miles* *5miles* *5miles*

Tortoise h_1 ——— h_2 ——— h_3 ——— h_4 ——— h_5

D.S.: Let us imagine that the race is 25 miles long with a check point at every 5 miles, so that $a_1 \rightarrow a_2$ = 5 miles., $h_1 \rightarrow h_2$ = 5 miles, and so on. Let's also imagine Achilles runs 10 mph and the Tortoise runs 5 mph, since Achilles is faster. Knowing this, we can determine that it will take Achilles 30 minutes to run between each point…

Therefore, 30 minutes into the race, Achilles would be at point a_2 and the Tortoise would be between h_1 and point h_2. An hour into the race, Achilles would be at point a_3 and the Tortoise would be at point h_2. Note $a_3 = h_2$. An hour and a half into the race, Achilles would have passed the Tortoise and be at point a_4, while the Tortoise is between the point h_2 and the point h_3.. Though Zeno's conclusion that the quicker will never pass the slower (if given a head start) may have been valid during the fifth century B.C., it does not stand true to today's experiences.

This and other essays informed us about the main difficulty students encounter and that is the absence of the converging sequence in their essay interpretations. It was not clear whether the issue is of algebraic nature, that is, whether absence of algebraic proficiency is responsible for that absence of understanding, or it is a question of conceptual understanding. On the other hand, it is clear that the student took equally spaced "check points" as the basis of analysis rather than points whose distance to the preceeding diminishes. This might indicate a problem solving issue of not understanding that these distances indeed should decrease. To distinguish between the possibilites, the next version of the assignment had concrete numbers assigned to the pertinent concepts. The essays also revealed absence of clear understanding of the phrase "head start", and the assignment was refined with respect to this issue as well.

Assignment 3

You have read about the story of Achilles and the Tortoise in Essay 1. Consider the following. Both, Achilles and the Tortoise start the race at the same time, but Tortoise starts 100m ahead of Achilles. Achilles starts at the point S, while Tortoise starts 100m ahead of S. Let the speed of Achilles be 10 m/sec and that of the Tortoise be 1 m/sec.

Achilles a_1 a_2 a_3 a_4 a_5 ………

Tortoise | t_1 t_2 t_3 t_4 ………

S

1. Fill in the dotted lines in the following table:

Position of Achilles	*Position of Tortoise*	*Distance of Achilles from S*	*Distance of Tortoise from S*	*Time in Seconds*
a_1	t_1	………	………	………
a_2	t_2	………	………	………
a_3	t_3	………	………	………

2. Are all of Zeno's conditions satisfied in your calculations above? Explain

The response of the same student as in Assignment 2 is quoted below.

Let's imagine that Achilles and the Tortoise start the race at the same time, but the Tortoise starts 100m ahead of Achilles. Achilles starts at the point S (a_1), while the Tortoise starts 100 m ahead of S (t_1). Let's also imagine Achilles runs 10m/sec and the Tortoise runs 1m/sec. Knowing this, we can determine the exact position of Achilles and the Tortoise over a period of time.

Achilles a_1 a_2 a_3 a_4 a_5 ………

Tortoise | t_1 t_2 t_3 t_4 ………

S

Position of Achilles	*Position of Tortoise*	*Distance of Achilles from S*	*Distance of Tortoise from S*	*Time in Seconds*
a_1	t_1	*0m*	*100m*	*0*
a_2	t_2	*100m*	*110m*	*10*
a_3	t_3	*200m*	*120m*	*20*
a_4	t_4	*300m*	*130m*	*30*

At a_1, we know Achilles is 0m from S because this is his starting point. We know the Tortoise is 100m from S at t_1, because this is his starting point. [Note: only 0 seconds have elapsed in the first row because this is the beginning of the race]. Since we know $a_2=t_1$ (refer to first diagram), we can conclude that Achilles is 100m from S at a_2 because the Tortoise began at t_1, which was 100m ahead of S. If Achilles is 100m from S at a_2, 10 seconds must have passed because he runs 10m/sec [Solve by: 100m / 10 m = 10 sec]. Since we know the Tortoise runs 1 m/sec, he must be 110m from S at t_2 [100m + (10 sec * 1m) = 110m].

To find the distance of Achilles from S at a_3, we must find a *pattern between a_1 and a_2. Since he was 0m away from S* *at a_1 and 100m away from S at a_2, we notice that it is going in increments of 100.* So at a_3 Achilles should be 200m from S [100m + 100 = 200m]. If Achilles is 200 m from S, we know 20 seconds must have passed [200m /10m = 20 sec]. That means the Tortoise is 120m from S at t_3 [100m + (20 sec * 1m) = 120m]. If you were to continue the pattern, *30 seconds into the race, Achilles would be 300m* from S at a_4 and the Tortoise would be 130m from S at t_3. Therefore it is impossible for the Tortoise to win the race. Achilles would always win because for every 10 seconds, he increases his position by 10m every 10 seconds.

We see that certain elements of understanding have appeared, for example, an attempt at the coordination of the actions of Achilles with those of the Tortoise – one of the central mental acts in understanding the nature of the paradox. At the same time we see that student still can not push the analysis beyond uniform motion because of the fundamental error in observing the patterns of the increase of the distance without taking into account sufficient number of cases. This error reveals a good student whose way of reasoning was affected by absence of clarity on the meaning of "patterns". It maybe reinforced by the basic conception of a uniform movement of the race's participants, whose irregularity is imposed somewhat artificially by the Zeno condition. How many cases are necessary to consider before one is ready to generalize? – is the question for future investigations, which should be done in the framework of strategy of hints mentioned by a mathematics teacher from NYC, Mark Saul (Saul, 1995). What is most important is that with all the technical improvement we still didn't have a single procedural evidence of the converging sequence of decreasing distances present in the thinking of students, our main object of investigation. Hence, we have not yet found the lower level of the ZPD of students to enable the proper integration of both levels.The 4th essay was assigned in the second half of the semester, when more mathematical tools became available to students. It is re-designed according to the analysis above.

Assignment 4 (November, 2004)

Suppose Achilles runs ten times as fast as the tortoise and gives him a hundred yards head start. In order to win the race Achilles must first make up for his initial handicap by running a hundred yards; but when he has done this and has reached the point where the tortoise started, the animal has had time to advance ten yards. While Achilles runs these ten yards, the tortoise gets one yard ahead; when Achilles has run this yard, the tortoise is a tenth of a yard ahead; and so on.

Graph the distance traveled by Achilles as a function of time and on the same grid, also the distance traveled by Tortoise as a function of time. Project the sequence of distances onto the vertical axis. Is this an increasing sequence of numbers, which is bounded above (Hint: is the length of the racecourse finite?) By Axiom 1, this sequence must have a smallest number greater than every term of the sequence. How will you prove that this number is the limit of the sequence?

The response of the same student as in Assignment 2 and 3 is given below.

For example, let's imagine Achilles runs ten times as fast as the tortoise and gives him a hundred yards start. The projection of the race would look as follows:

Time	*Distance Ran in Yards*	
	Achilles	*Tortoise*
...	*0*	*100*
t_1	*100*	*110*
t_2	*110*	*111*
t_3	*111*	*111.1*
t_4	*111.1*	*111.11*
t_5	*111.11*	*111.111*

From the graph, we can determine that this is an increasing sequence of numbers because the distance ran at t_{n+1} is always greater than t_n. To find out if this sequence is bounded above, we must look at Axiom 1. Axiom 1 states, "If M is a set of numbers that is bounded above, then either there is a largest number in M or there is a smallest number that is larger than every number in M." Since the length of the racecourse is infinite, there isn't a largest number. On the other hand, there is a smallest number greater than every term of the sequence, although it may not be as clear to some people as to what it is. In past problems it was easy to determine the smallest number greater than every term of the sequence [its clear that the sequence {½, ⅔, ¾, ⅘, ⅚, …} is bounded above at 1]. However, the sequence in this particular problem is {…, 111.1, 111.11, 111.111, …}; which can also be written as {…, $111.\underline{1}$}. In situations like this we can implicitly use infinite sums.

$111\ {}^1\!/_{10} + 111\ {}^1\!/_{100} + 111\ {}^1\!/_{1000} + 111\ {}^1\!/_{10000} + \ldots = 111.\underline{1}$

Observe that as you add more and more terms, the partial sums become closer and closer and closer to $111\,{}^1\!/_9$. Therefore, we can say $111\,{}^1\!/_9$ is the smallest number greater than every term of the sequence (and also the limit of the sequence).

To prove that $111\,{}^1\!/_9$ is the limit of the sequence, we can apply the geometric definition of convergence of a sequence. This definition states, "The number L is the limit of the infinite sequence a if and only if for any two equally spaced horizontal lines with the point (0, L) between them, there are at most finitely many points outside the two horizontal lines and infinitely many points between the lines." For example, if you were to draw 2 horizontal lines at 111 and 111⅖ (both lines being ⅕ away from 111⅕), there will be at most finitely many terms outside of the horizontal lines and infinitely many between the lines (*note: the horizontal lines can be drawn anywhere as long as it contains the limit). Hence, 111⅕ is the limit of the sequence.

Discussion

It is clear that the last assignment had had a strong impact upon the particular student enabling him to make the connection between the sequential content of A&T paradox and the definition of the limit of a sequence. Coordination of the two situations is excellent, showing all necessary connections, except for the final interpretation of the definition which misses the universal quantifier. In terms of the inquiry technique, the teacher-researcher was able to facilitate a significant moment of understanding in the student's mental apparatus relative to the problem, in terms of ZPD, she was able to outline the full scope of ZPD of the student starting at the hint leading to grasping the decreasing aspect of the distances to full understanding. A significant step in the construction of understanding was also the recognition of interaction of distances and time intervals between Achilles and Tortoise. We see in the example above the process of coordination of the classroom experience with the theoretical framework of ZPD of Vygotsky. The guide of this coordination was the very definition of ZPD as the conceptual distance between what student can do on his/her own and what she/he can grasp with the help of the intsructor or a properly chosen peer group. Having introduced during the course the appropriate definitions of the limit of a converging sequnce as the "scientific" concept, the teacher-researchers were searching for such a hint of intuitive nature, which would allow the student to construct the understanding reaching the level of the definition. We found out that the correct hint, and hence the extent of ZPD in the case of the student whose excerpts we discussed, *was the indication of the systematic decrease in the length of distances run by Achilles.*

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