Students Using a Mathematica Learning Project Work to Conceptualise the Derivative

Kristie Naidoo and Ramu Naidoo

Durban University of Technology, Durban, South Africa

This study is based on a constructivist approach to the learning of the concept of the derivative by discovery and by self-pacing. To determine whether the new learning and teaching environment had an impact on the students' understanding of the derivative, two groups, each consisting of 34 students, comprised the control and experimental groups respectively. The experimental group participated in a Mathematica Learning Project whilst the control group was taught traditionally. Both groups were tested. It was envisaged that Mathematica Learning project may minimize the cognitive overload experienced by students during a traditional lecture. The project work in the laboratory formed part of the assessment for the experimental group. To evaluate students' responses, errors made by students during the project and the paper-pencil test were analysed. Findings revealed a greater number of structural errors in the control group as compared to the experimental group. Further the experimental group exhibited more deep structures than surface structures whilst the traditional group exhibited more superficial structures than deep structures.

Introduction

Two groups of 34 students, one the experimental group and the other the control group, participated in this project. Both groups were made up of homogenous students with varying mathematical ability. All students had to satisfy the University requirement of a minimum symbol either 'E' on the higher grade or 'D' on the standard grade in their matriculation examination results. The control group attended traditional lectures that consisted of chalk and talk lessons in a University lecture theatre. They were taught by a team member of the Mathematics Research Group established at the University. The students were supported by a weekly tutorial where students would interact with a tutor in areas of concern. The experimental group was oriented to *Mathematica* software in a laboratory environment in four two-hourly sessions by members of the Mathematics Research Group. During laboratory sessions students were able to collaborate with members of their group and members of the Research Group. After determining that there was reasonable competency with syntax and other computer related aspects, the students were asked to complete the compulsory mathematics project from the Department of Mathematics at the University of Technology. This project was also part of their course fulfillment requirements. This ensured that students would find the experience beneficial and work to the best of their ability since it also contributed to their course mark. Immediately after the project was completed both the experimental group and control group completed Orton's (1983) tasks on differentiation.

Findings by Naidoo (1996) deduced that, in a traditional classroom, first year mathematics students study by rules. They do not enjoy mathematics and are de-motivated. Lecturers tend to teach mechanistically and do standard type solutions to standard type problems. He drew attention to the fact that mathematics at the University of Technology (previously Technikon) level is not a specialist subject. This contributes to the "poor" understanding of critical concepts that are essential for extended learning – a type of understanding that is needed to support an increasingly technological world. Consequently the time and attention given to study mathematics is limited.

Naidoo and Naidoo (2007) found that teaching and learning using the computer laboratory gave a measure of success.

Although students were performing better, they were still making errors. Some of these errors were: (i) inability to conclude that sequences converge (ii) problems with rate of change of a curve and (iii) inability to interpret symbols. Studies done on Calculus Reform (Silverberg, 2004) indicate a measure of success. Silverberg was unable to draw firm conclusions about achievement in mathematics grades when comparing the traditional and reform cohorts. In contrast, his analysis of follow-up grades produced the most compelling results. The comparison of grades in follow-on courses such as physics and engineering science showed significant improvements between cohorts. After some time the reform group performed better in these courses.

Blended Learning environment which combined face-to-face sessions together with computer interaction (similar to Owston et al., 2006) was used to compliment traditional lectures. Pre-knowledge concepts that are underdeveloped in the traditional learning environment were tested and manipulated as objects in a Blended Learning environment. For example, function, graph, and rate of change can be processed in simple and concrete ways and then reconceptualised. Students are able to explore pre-knowledge concepts and shape their learning. These include function, gradient and ratio. Two categories of software, programmable microworlds and expressive tools where such transformation is significant were identified (Noss & Hoyles, 2004). Mackie (2002) changed the emphasis of teaching and learning of calculus from techniques and routine symbolic manipulation towards higher level cognitive skills that focus on concepts and problem solving. She found that it encouraged students to become reflective deep learners.

Project Work in Mathematics

Vithal (2004) found projects or project work to form a "progressive" approach to mathematics education and advocates more "open-ended", "problem-centered" activities in which learners are given greater independence in their learning, in contexts relevant to them. In terms of the outcomes based approach to learning, project work is extensively used as an assessment strategy in modern South African Schools. Not much research exists to test the effect and use of project work. In countries like Scandinavia and Denmark project work had been introduced for decades.

Wurnig (2004), found significant changes in the learning process when using project work in mathematics problem solving. Some of these were: (i) the learning process is more student oriented; (ii) the lecturer is not the only source of knowledge; (iii) there are more mathematical discussions among students; (iv) the computer is an additional expert in the learning situation; (v) there is an emphasis on problem solving and application oriented mathematics. Using technology like Mathematica, students can explore and model mathematics concepts. When hands-on computer activities are used students benefit in that the computer is able to sketch graphs and carrying out manipulations frees the student to concentrate on the concepts.

The Derivative and The Mathematica Project

We review theoretical issues in the literature which explore some of the concepts and processes associated with differentiation. The derivative can be seen as a concept which is built from other concepts. Particularly, the derivative can be seen as a function, a number if evaluated at a point, limit of the sequence of secant slopes or rate of change. Differentiation assumes the understanding of function or more generally a curve (not all curves can be formulated by a function). Each advanced concept in mathematics is based on elementary concepts and cannot be grasped without a solid and sometimes very specific understanding of these elementary concepts. Thus the concepts of advanced mathematics carry an intrinsic complexity. For example, students cannot grasp what is meant by a differential equation or interpret its solution unless they have understood the derivative concept and not just the techniques of differentiation. Mathematicians explain the derivative using pre-concepts such as elementary algebra, rates of change, limits and infinity and tangents. The network or sequence leads to interrelated ideas, each idea integrating some of the more elementary ones into an added structure. It is precisely the complexity of concepts that makes differentiation difficult for students to grasp.

There is a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived. The term 'concept image' describes the total cognitive structure that is associated with the concept. Tall & Vinner (1981) indicate that the concept image includes all mental pictures and associated properties and processes. In coming to understand mathematical concepts at school, students evolve mental pictures at a concrete level. For example, to understand rate of change students evoke pictures of a moving car. The mental pictures which served the students well at school level may now become an impediment. Bruner (1986) suggested that iconic processing limited ideas and urged a movement onto the symbolic level. The student with an inadequate concept image may find such a development difficult to achieve. The concept image of the limit may evoke a mental frame of a chord (secant) tending to a tangent which is a form of a metaphor as described by Oertman (2003). A qualitative theoretical

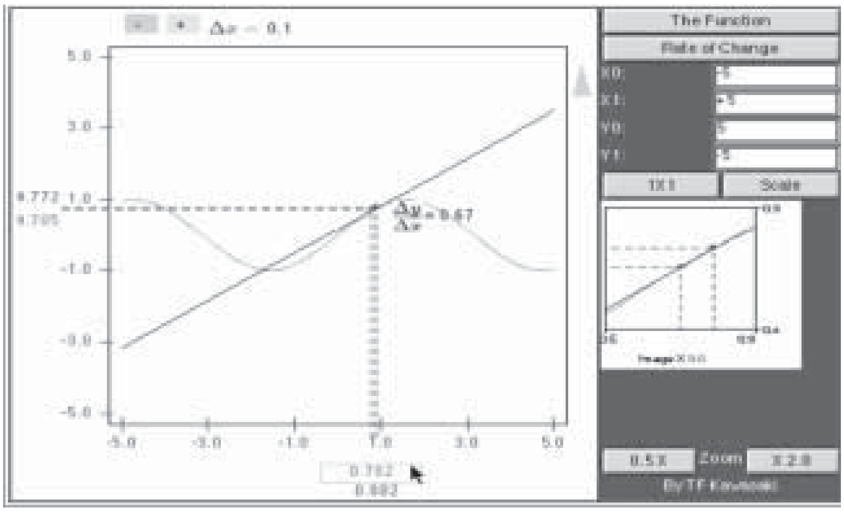

Fig. 1. Software used to zoom over a small interval on a curve

framework was constructed by Naidoo (1996) from the analysis of errors in arithmetic by Donaldson (1963), cognitive frame theory by Davis (1984) and the modified Orton's (1983) tasks.

David Tall (1996) used the "local straightness" of the graph as his "good" cognitive root to build calculus. His Graphic Calculus software enabled the student to magnify a portion of the graph to observe the straightness by tracing the gradient numerically along the graph. Additional software allowed the student to point the mouse at a given place in the plane and draw a line segment of the given gradient. An approximate solution could be constructed physically and visually by sticking segments from end to end. This is a means of encouraging deep approaches to learning. The student is motivated further by adding reality to his solution.

The derivative theory can be taught using the traditional method where the student is shown that a series of secants form a converging sequence. This is an overload for the average learner since knowledge of the secant, tangent and other geometrical knowledge is involved.

Zoom Function and The Derivative

The zoom graph method is a computer lab experience where the curve is approximated to a straight line. It is effective because the student is dealing only with gradients of a straight line. When the domain intervals are very small the curve can be approximated as a straight line. Instead of using secants we zoom to get a straight line. Here we establish the idea of the gradient of a curved graph. Using a suitable software the graph can be drawn and a part of the can be selected and magnified. The magnified part looks "straight". This method frees the student from cognitive overload. The student does not have to deal with tangents, secants and complex geometry. Tall (2002) agrees that calculus software should be programmed to assist the user to explore graphs with corners and wrinkled graphs. Figure 1 (adapted from Visual Calculus software programmed by Teresinha Kawasaki) shows how computer software can be used to zoom over a small interval on a curve. The rate of change can be found from both directions.

Using *mathematica* the students were able to deduce the derivative by a secant converging to a tangent and from the rate of change of a straight line. In this way students were able to use multiple representations to deal with the derivative concept. Traditional learning and blended learning is combined in a flexible way (Gray, 2006) using a combination of options at each stage of the delivery. Students were able to explore concepts that could not be explained easily using only traditional approaches.

Analysis of Data

The data from the project work consisted of student project protocols from the project tasks.

Table 1 shows responses categorized as deep, intermediate and surface using specific criteria for three tasks. The criteria are presented along with the tasks below.

Project Task	Deep	Intermediate	Surface
A	8	14	6
B	9	15	4
C	8	16	4

Table: 1. Deep and surface learning in the mathematica project

TASK A: Find the limit of the following numerically and graphically: $\lim_{x \to 4}[(2-x^2)/(4-x)]$. Discuss your results. For the numeric values show explicitly whether the sequence is converging or diverging.

Deep: Response must include "sequences", "converges to a point" and "limit".

Intermediate: Response must include "sequences", and "converges to a point" but neglects "limit".

Surface: Response has "sequences" but does not mention convergence.

TASK B: Let $f(x) = 3x - 2x^2$

(i)Find the average rate of change of *f(x)* from $x = 0.5$ to $x = 0.9$. (ii) Find the equation of the corresponding secant line. (iii) Plot the graphs of *f(x)* and the secant line on the same axes. (iv) Repeat parts (i) to (iii) for $x = 0.5$ and $x = 0.51$. Explain what you observe. (v) Zoom in on the graph around the point $(0.5, f(0.5))$. Show your plot and explain what you observe about the two graphs in (iv). (vi) Re-plot the graph *f(x)* over the interval [0,1]. Now zoom in on the graph around the point $(0.5, f(0.5))$ until the graph looks like a straight line. Show your plot and explain how you can use this graph to estimate the slope of this line. (Hint: Move the mouse pointer to the line and click at two different points on it; then observe the first and second coordinates of the points you clicked on.)

Deep: Responses includes "function", "change in function: $f(x+\Delta x) - f(x)$", "line joining points (x, y) and $(x+\Delta x, y+\Delta y)$ on the graph" and "represents a secant line".

Intermediate: Responses includes "function", "change in function: $f(x+\Delta x) - f(x)$" but does not mention a secant line.

Surface: Does mention change in function, not able to indicate points (x, y) and $(x+\Delta x, y+\Delta y)$ on the graph and show that it represents a secant line.

TASK C: Let $f(x) = 3x - 2x^2$

(i) Find the instantaneous rate of change of *f(x)* at $x = 0.5$ using the definition of the derivative. (ii) Find the equation of the corresponding tangent line. (iii) Plot the graphs of *f(x)* and the tangent line on the same system of axes. Zoom in on the graph around the point $(0.5, f(0.5))$ until the two graphs are indistinguishable. How close did you have to get? (iv) Evaluate $[f(0.5+h) - f(0.5)]/h$. Explain how you can estimate the derivative of *f(x)* at 0.5.

Deep: Able to show "sequence of secants converge to a point to a tangent", and "slopes of secants converging to a slope of the tangent", and "$dy/dx = \lim_{\Delta x \to 0}[\Delta y/\Delta x]$", dy/dx is the slope of the tangent and instantaneous rate of change, and dy/dx is the average rate of change which is the slope of the secants.

Intermediate: Able to show "sequence of secants converge to a point to a tangent", and "slopes of secants converging to a slope of the tangent" but not responses "$dy/dx = \lim_{\Delta x \to 0}[\Delta y/\Delta x]$", dy/dx is the slope of the tangent and instantaneous rate of change, and dy/dx is the average rate of change which is the slope of the secants.

Surface: No distinction made between slopes of secants and tangents.

The experimental and control groups were compared on the basis of the errors they made on a set of tasks. Table 2 presents the typology of errors for the two groups. The tasks and the performance are discussed below.

Task 1 was based on the limit of an infinite geometric sequence. From a fixed point *P* on a circle (shown in a diagram) secants were drawn passing through various points Q_1, Q_2, ... Students were asked (1.1) how many such secants could be drawn and (1.2) what happens when *Q* gets closer and closer to *P*. The idea of the rotating secant was intended to relate to the approach to differentiation. This item would give evidence concerning the level of understanding of the tangent as a limit. 76% of the experimental group failed to make relationships. Table 1 shows that these errors were primarily structural errors. 6% of the students from the blended group made arbitrary errors; they failed to take the constraints into account. A larger percentage of the control group (94%) displayed structural errors. The frame 'sequences', 'tangent line' and 'limit' could not be retrieved. Vague answers like "as many as you want", "as many as possible" and "many of them" were characteristic of the incorrect responses. The required frames were sketchy and incomplete. Clearly students needed help in understanding the tangent as the limit of the set of secants. This task was also a sub-problem of Task 6. Comparing the responses from both the groups in each of these tasks revealed that there was a correlation between the poor performance in both the experimental group and the control group. This confirmed that an incomplete frame in one sub-frame would reflect incompleteness in another related frame.

Task 2, based on the rate of change from the straight line graph, was a problem about water flowing into an initially

Classification of items	Structural Errors		Executive Errors		Arbitrary Errors	
	Experimental	Control	Experimental	Control	Experimental	Control
Sequence Task 1.1	76	94	0	0	6	2
Limit Task 1.2 and Task 6	56	74	7	13	38	21
Average rate of change Task 3 (3.4), Task 4(4.4) and Task 4(4.5)	67	84	1	1	0	2
Rate of change straight line Tasks 2 and 3.5	79	79	0	1	0	0
Rate of change of a curve Tasks 4.1 – 4.5	24	71	0	3	0	0
Differentiation Tasks 5.1 – 5.2	12	32	26	32	0	2
Symbolism Tasks 7.1 - 7.7	4 2	48	86	90	12	9

Table: 2. Classification of errors

empty tank at constant rate, which was explicitly given as 2 units of depth per unit of time. The data was presented both in a table and in a coordinate graph. Students were asked what the rate of increase in the depth was when (2.1) $x = 2\ ½$ and (2.2) when $x = T$.

Task 3, also based on the rate of change from the straight line graph, presented the coordinate graph of $y = 3x - 1$ along with the equation. Students were asked the following: (3.1) What is the value of y when $x = a$? [a is a real number] (3.2) What is the value of y when $x = a + h$? [h is any increment] (3.3) What is the increase in y as x increases from a to $a + h$? (3.4) What is the rate of increase of y as x increases from a to $a + h$? (3.5) What is the rate of increase of y at $x = 2\ ½$? and at $x = X$?

Task 2 and 3.5 were grouped for analysis. In Task 2, at $x = 2½$, a large number of the subjects gave the value of y, which is 5, instead of the rate. It was apparent that both experimental and control groups did not grasp the meaning of the explicit mention of the rate as 2 units per unit of time. At the general point x = T in task 3.5, the responses were worse. There was a significant misunderstanding between the rate of change at a point and the y-value at that point. It is also possible that the students had no conception of rate of change at all. This is why they worked out the y-value, given the x-value. A fairly large number of structural errors were recorded. This represented 79% of the sample in each category. Clearly students were unable to retrieve the frame 'a tank being filled with water', ' a straight line graph with gradient 2' and 'rate of change equal to gradient'. In particular the frame 'straight line graph' was incomplete. Within this frame the algebraic sub-frame was also not developed. This task represented a real world problem. Another explanation that could be afforded is that the students were not subject to real world problems during their lecture and tutorial sessions. These responses represent the experience of the students, a type of experience that is characterized by doing problems by "drill" or using the mechanistic approach.

Task 4, concerned with finding the average rate of change from a curve, presented the coordinate graph of $y = 3x^2 + 1$ along with the equation. Students were asked the following: (4.1) What is the value of y when $x = a$? [a is a real number] (4.2) What is the value of y when $x = a + h$? [h is any increment] (4.3) What is the increase in y as x increases from a to $a + h$? (4.4) What is the average rate of change in the x-interval a to $a + h$? (4.5) Can you use the result in 4.4 to obtain the rate of change of y at $x = 2½$? At $x = T$? If so, how?

24% of the experimental group and 71% of the control group made structural errors in this task. A greater percentage of students from the experimental group were able to retrieve the frame required for the solution of this task 'the average rate of change can be calculated from any two points irrespective of the curve'. This shows that their interaction with the computer had reinforced this frame. The students from the control group were baffled. An interesting observation was that this is a typical real world problem encountered in engineering.

Task 5 dealt with differentiation and contained the following questions: (5.1) What is the formula for the rate of change for the equation $y = x^n$? [n is an element of the natural numbers] (5.2) What is the rate of change formula for each of the following equations? (a) $y = 3x^3$ (b) $y = 4$ (c) $y = 2/x^2$?

12% of the experimental group recorded structural errors and 32% of the control group recorded structural errors. 26% of the experimental group made executive errors and 32% of the control group made executive errors. Students have lost track of the algorithm that they were trying to use. Davis (1984) refers to this as a control error. The student has memorized a rule he/she has been following or they behave in a certain way because they know from experience that this is an effective or appropriate way to tackle the problem. The majority of the students were able to employ the mechanistic methods that were needed to solve the task. It is clear from the data that students have mastered the "rules" required to undertake this task. This confirms that frame 'rules for differentiation' were easily accessible to these students.

Task 6 was based on differentiation as a limit. It presented a diagram used in engineering mathematics to introduce the definition of the derivative, viz. $dy/dx = \lim_{h\to 0}[y(x+h) - y(x)]/h$, where y is any function and h is an increment in x. The students were asked: (6.1) At which point or points of the graph does the formula measure the rate of change? (6.2) Explain why the formula defines this rate of change?

The combined data for this task and task 1.2 shows that 56% of the experimental group made structural errors. A high percentage made arbitrary errors (38%). It is evident that these students did not understand the definition for the derivative. 74% of the control group made structural errors and 21% made arbitrary errors. The percentage of arbitrary errors is less than that of the experimental group. This can be attributed to the fact that a single answer response was needed for this task and it became a problem to classify a wrong answer, like Q, for instance. The majority of the students were unable to retrieve the frame 'instantaneous rate of change'. The 'congruent motive-strategy package' described by Biggs (1986) is prevalent here. A larger percentage of the experimental group gave a correct response. They were able to show sound reasoning based on understanding.

Task 7 asked students to explain the meaning of the following symbols: (7.1) δx; (7.2) δy; (7.3) dy/dx; (7.4) dx; (7.5) dy; (7.6) dy/dx; (7.7) What is the relationship between $\delta y/\delta x$ and dy/dx?

The symbols that were given represented standard notation used in elementary calculus and those that must be understood by students. 42% of the experimental group made structural errors and 48% of the control group exhibited structural errors. It showed that a large percentage of the students were unable to connect the various symbols meaningfully. Clearly these symbols were confusing to both groups. These may not have been explained adequately in the lectures or the frame 'symbolic images' is lacking in both groups. A number of students were able to say that δx and δy represented small increments in the x-direction and y-direction respectively. It would appear that students have met these symbols before. However students were not able to explain the quotient $\delta y/\delta x$ correctly. The symbols dx and dy were misinterpreted. Students could not make sense of these symbols if they were not written as a quotient dy/dx.

Conclusion

Student errors in elementary differential calculus can be minimized by using a blended teaching approach. Many factors need to be considered when referring to students understanding of differentiation. These include: (i) weak preknowledge frames (ii) reliance on algorithmic means to solve problems (iii) errors made by students and (iv) symbolism in elementary calculus. In the learning of elementary calculus it is essential that a mechanistic application of a set of rules is not sufficient, rather the synthesis of the appropriate mental frames is needed to represent concepts and the procedures necessary to seek solutions.

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